# CHAPTER 2

## Differential Calculus of Functions of One Variable

IN THIS CHAPTER we study the differential calculus of functions of one variable.

SECTION 2.1 introduces the concept of function and discusses arithmetic operations on functions, limits, one-sided limits, limits at $\pm\infty$, and monotonic functions.

SECTION 2.2 defines continuity and discusses removable discontinuities, composite functions, bounded functions, the intermediate value theorem, uniform continuity, and additional properties of monotonic functions.

SECTION 2.3 introduces the derivative and its geometric interpretation. Topics covered include the interchange of differentiation and arithmetic operations, the chain rule, one-sided derivatives, extreme values of a differentiable function, Rolle's theorem, the intermediate value theorem for derivatives, and the mean value theorem and its consequences.

SECTION 2.4 presents a comprehensive discussion of L'Hospital's rule.

SECTION 2.5 discusses the approximation of a function $f$ by the Taylor polynomials of $f$ and applies this result to locating local extrema of $f$. The section concludes with the extended mean value theorem, which implies Taylor's theorem.

## 2.1 FUNCTIONS AND LIMITS

In this section we study limits of real-valued functions of a real variable. You studied limits in calculus. However, we will look more carefully at the definition of limit and prove theorems usually not proved in calculus.

A rule $f$ that assigns to each member of a nonempty set $D$ a unique member of a set $Y$ is a *function from $D$ to $Y$*. We write the relationship between a member $x$ of $D$ and the member $y$ of $Y$ that $f$ assigns to $x$ as

$$y = f(x).$$

The set $D$ is the *domain* of $f$, denoted by $D_f$. The members of $Y$ are the possible *values* of $f$. If $y_0 \in Y$ and there is an $x_0$ in $D$ such that $f(x_0) = y_0$, we say that $f$

*attains* or *assumes* the value $y_0$. The set of values attained by $f$ is the *range* of $f$. A *real-valued function of a real variable* is a function whose domain and range are both subsets of the reals. Although we are concerned only with real-valued functions of a real variable in this section, our definitions are not restricted to this situation. In later sections we will consider situations where the range or domain, or both, are subsets of vector spaces.

**Example 2.1.1** The functions $f$, $g$, and $h$ defined on $(-\infty, \infty)$ by

$$f(x) = x^2, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = e^x$$

have ranges $[0, \infty)$, $[-1, 1]$, and $(0, \infty)$, respectively. ■

**Example 2.1.2** The equation

$$[f(x)]^2 = x \tag{1}$$

does not define a function except on the singleton set $\{0\}$. If $x < 0$, no real number satisfies (1), while if $x > 0$, two real numbers satisfy (1). However, the conditions

$$[f(x)]^2 = x \quad \text{and} \quad f(x) \geq 0$$

define a function $f$ on $D_f = [0, \infty)$ with values $f(x) = \sqrt{x}$. Similarly, the conditions

$$[g(x)]^2 = x \quad \text{and} \quad g(x) \leq 0$$

define a function $g$ on $D_g = [0, \infty)$ with values $g(x) = -\sqrt{x}$. The ranges of $f$ and $g$ are $[0, \infty)$ and $(-\infty, 0]$, respectively. ■

It is important to understand that the definition of a function includes the specification of its domain and that there is a difference between $f$, the *name* of the function, and $f(x)$, the *value* of $f$ at $x$. However, strict observance of these points leads to annoying verbosity, such as "the function $f$ with domain $(-\infty, \infty)$ and values $f(x) = x$." We will avoid this in two ways: (1) by agreeing that if a function $f$ is introduced without explicitly defining $D_f$, then $D_f$ will be understood to consist of all points $x$ for which the rule defining $f(x)$ makes sense, and (2) by bearing in mind the distinction between $f$ and $f(x)$, but not emphasizing it when it would be a nuisance to do so. For example, we will write "consider the function $f(x) = \sqrt{1 - x^2}$," rather than "consider the function $f$ defined on $[-1, 1]$ by $f(x) = \sqrt{1 - x^2}$," or "consider the function $g(x) = 1/\sin x$," rather than "consider the function $g$ defined for $x \neq k\pi$ ($k$ = integer) by $g(x) = 1/\sin x$." We will also write $f = c$ (constant) to denote the function $f$ defined by $f(x) = c$ for all $x$.

Our definition of function is somewhat intuitive, but adequate for our purposes. Moreover, it is the working form of the definition, even if the idea is introduced more rigorously to begin with. For a more precise definition, we first define the *Cartesian product* $X \times Y$ of two nonempty sets $X$ and $Y$ to be the set of all ordered pairs $(x, y)$ such that $x \in X$ and $y \in Y$; thus,

$$X \times Y = \left\{(x, y) \,\middle|\, x \in X, y \in Y\right\}.$$

A nonempty subset $f$ of $X \times Y$ is a *function* if no $x$ in $X$ occurs more than once as a first member among the elements of $f$. Put another way, if $(x, y)$ and $(x, y_1)$ are in $f$, then $y = y_1$. The set of $x$'s that occur as first members of $f$ is the *domain* of $f$. If $x$ is in the domain of $f$, then the unique $y$ in $Y$ such that $(x, y) \in f$ is the *value of $f$ at $x$*, and we write $y = f(x)$. The set of all such values, a subset of $Y$, is the *range of $f$*.

## Arithmetic Operations on Functions

**Definition 2.1.1** If $D_f \cap D_g \neq \emptyset$, then $f+g$, $f-g$, and $fg$ are defined on $D_f \cap D_g$ by

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x),$$

and

$$(fg)(x) = f(x)g(x).$$

The quotient $f/g$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

for $x$ in $D_f \cap D_g$ such that $g(x) \neq 0$. ■

**Example 2.1.3** If $f(x) = \sqrt{4-x^2}$ and $g(x) = \sqrt{x-1}$, then $D_f = [-2, 2]$ and $D_g = [1, \infty)$, so $f+g$, $f-g$, and $fg$ are defined on $D_f \cap D_g = [1, 2]$ by

$$(f+g)(x) = \sqrt{4-x^2} + \sqrt{x-1},$$
$$(f-g)(x) = \sqrt{4-x^2} - \sqrt{x-1},$$

and

$$(fg)(x) = (\sqrt{4-x^2})(\sqrt{x-1}) = \sqrt{(4-x^2)(x-1)}. \tag{2}$$

The quotient $f/g$ is defined on $(1, 2]$ by

$$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{4-x^2}{x-1}}.$$

Although the last expression in (2) is also defined for $-\infty < x < -2$, it does not represent $fg$ for such $x$, since $f$ and $g$ are not defined on $(-\infty, -2]$. ■

**Example 2.1.4** If $c$ is a real number, the function $cf$ defined by $(cf)(x) = cf(x)$ can be regarded as the product of $f$ and a constant function. Its domain is $D_f$. The sum and product of $n$ $(\geq 2)$ functions $f_1, \dots, f_n$ are defined by

$$(f_1 + f_2 + \cdots + f_n)(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

and

$$(f_1 f_2 \cdots f_n)(x) = f_1(x) f_2(x) \cdots f_n(x) \tag{3}$$

on $D = \bigcap_{i=1}^n D_{f_i}$, provided that $D$ is nonempty. If $f_1 = f_2 = \cdots = f_n$, then (3) defines the $n$th *power of* $f$:

$$(f^n)(x) = (f(x))^n.$$

From these definitions, we can build the set of all *polynomials*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

starting from the constant functions and $f(x) = x$. The quotient of two polynomials is a *rational function*

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m} \quad (b_m \neq 0).$$

The domain of $r$ is the set of points where the denominator is nonzero. ■

## Limits

The essence of the concept of limit for real-valued functions of a real variable is this: If $L$ is a real number, then $\lim_{x \to x_0} f(x) = L$ means that the value $f(x)$ can be made as close to $L$ as we wish by taking $x$ sufficiently close to $x_0$. This is made precise in the following definition.

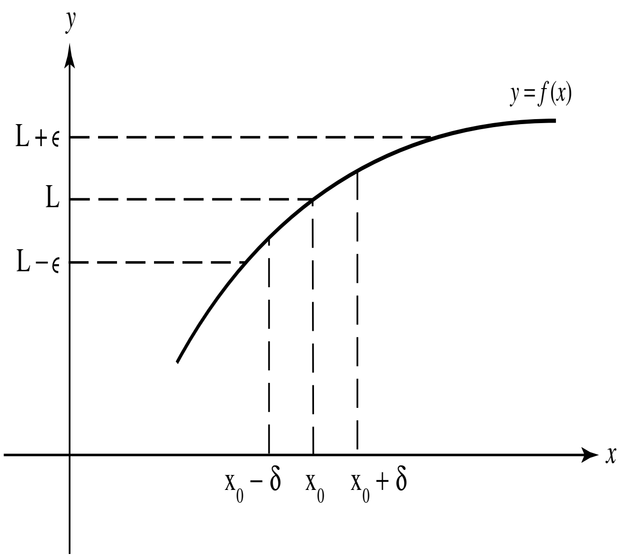


**Figure 2.1.1**

**Definition 2.1.2** We say that $f(x)$ *approaches the limit* $L$ *as* $x$ *approaches* $x_0$, and write

$$\lim_{x \to x_0} f(x) = L,$$

if $f$ is defined on some deleted neighborhood of $x_0$ and, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \tag{4}$$

if

$$0 < |x - x_0| < \delta. \tag{5}$$

Figure 2.1.1 (page 33) depicts the graph of a function for which $\lim_{x \to x_0} f(x)$ exists. ■

**Example 2.1.5** If $c$ and $x$ are arbitrary real numbers and $f(x) = cx$, then

$$\lim_{x \to x_0} f(x) = cx_0.$$

To prove this, we write

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0|.$$

If $c \neq 0$, this yields

$$|f(x) - cx_0| < \epsilon \tag{6}$$

if

$$|x - x_0| < \delta,$$

where $\delta$ is any number such that $0 < \delta \le \epsilon/|c|$. If $c = 0$, then $f(x) - cx_0 = 0$ for all $x$, so (6) holds for all $x$. ■

The next theorem says that a function cannot have more than one limit at a point.

**Theorem 2.1.3** *If* $\lim_{x\to x_0} f(x)$ *exists, then it is unique; that is, if*

$$\lim_{x\to x_0} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to x_0} f(x) = L_2, \tag{7}$$

*then* $L_1 = L_2$.

**Proof** Suppose that (7) holds and let $\epsilon > 0$. From Definition 2.1.2, there are positive numbers $\delta_1$ and $\delta_2$ such that

$$|f(x) - L_i| < \epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_i, \quad i = 1, 2.$$

If $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta. \end{aligned}$$

We have now established an inequality that does not depend on $x$; that is,

$$|L_1 - L_2| < 2\epsilon.$$

Since this holds for any positive $\epsilon$, $L_1 = L_2$. □

Definition 2.1.2 is not changed by replacing (4) with

$$|f(x) - L| < K\epsilon, \tag{8}$$

where $K$ is a positive constant, because if either of (4) or (8) can be made to hold for any $\epsilon > 0$ by making $|x - x_0|$ sufficiently small and positive, then so can the other (Exercise 5). This may seem to be a minor point, but it is often convenient to work with (8) rather than (4), as we will see in the proof of the following theorem.

## A Useful Theorem about Limits

**Theorem 2.1.4** *If*

$$\lim_{x\to x_0} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to x_0} g(x) = L_2, \tag{9}$$

*then*

$$\lim_{x\to x_0} (f+g)(x) = L_1 + L_2, \tag{10}$$

$$\lim_{x\to x_0} (f-g)(x) = L_1 - L_2, \tag{11}$$

$$\lim_{x\to x_0} (fg)(x) = L_1 L_2, \tag{12}$$

*and, if* $L_2 \neq 0$, (13)

$$\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}. \tag{14}$$

**Proof** From (9) and Definition 2.1.2, if $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$|f(x) - L_1| < \epsilon \tag{15}$$

if $0 < |x - x_0| < \delta_1$, and a $\delta_2 > 0$ such that

$$|g(x) - L_2| < \epsilon \tag{16}$$

if $0 < |x - x_0| < \delta_2$. Suppose that

$$0 < |x - x_0| < \delta = \min(\delta_1, \delta_2), \tag{17}$$

so that (15) and (16) both hold. Then

$$\begin{aligned} |(f \pm g)(x) - (L_1 \pm L_2)| &= |(f(x) - L_1) \pm (g(x) - L_2)| \\ &\le |f(x) - L_1| + |g(x) - L_2| < 2\epsilon, \end{aligned}$$

which proves (10) and (11).

Successive applications of the various parts of Theorem 2.1.4 permit us to find limits without the $\epsilon$–$\delta$ arguments required by Definition 2.1.2.

**Example 2.1.7** Use Theorem 2.1.4 to find

$$\lim_{x\to 2} \frac{9-x^2}{x+1} \quad \text{and} \quad \lim_{x\to 2}(9-x^2)(x+1).$$

**Solution** If $c$ is a constant, then $\lim_{x\to x_0} c = c$, and, from Example 2.1.5, $\lim_{x\to x_0} x = x_0$. Therefore, from Theorem 2.1.4,

$$\begin{aligned}\lim_{x\to 2}(9-x^2) &= \lim_{x\to 2} 9 - \lim_{x\to 2} x^2\\ &= \lim_{x\to 2} 9 - (\lim_{x\to 2} x)^2\\ &= 9 - 2^2 = 5,\end{aligned}$$

and

$$\lim_{x\to 2}(x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Therefore,

$$\lim_{x\to 2} \frac{9-x^2}{x+1} = \frac{\lim_{x\to 2}(9-x^2)}{\lim_{x\to 2}(x+1)} = \frac{5}{3}$$

and

$$\lim_{x\to 2}(9-x^2)(x+1) = \lim_{x\to 2}(9-x^2)\lim_{x\to 2}(x+1) = 5\cdot 3 = 15. \qquad \blacksquare$$

## One-Sided Limits

The function

$$f(x) = 2x \sin\sqrt{x}$$

satisfies the inequality

$$|f(x)| < \epsilon$$

if $0 < x < \delta = \epsilon/2$. However, this does not mean that $\lim_{x\to 0} f(x) = 0$, since $f$ is not defined for negative $x$, as it must be to satisfy the conditions of Definition 2.1.2 with $x_0 = 0$ and $L = 0$. The function

$$g(x) = x + \frac{|x|}{x}, \quad x \neq 0,$$

can be rewritten as

$$g(x) = \begin{cases} x + 1, & x > 0, \\ x - 1, & x < 0; \end{cases}$$

hence, every open interval containing $x_0 = 0$ also contains points $x_1$ and $x_2$ such that $|g(x_1) - g(x_2)|$ is as close to 2 as we please. Therefore, $\lim_{x\to x_0} g(x)$ does not exist (Exercise 26).

Although $f(x)$ and $g(x)$ do not approach limits as $x$ approaches zero, they each exhibit a definite sort of limiting behavior for small positive values of $x$, as does $g(x)$ for small negative values of $x$. The kind of behavior we have in mind is defined precisely as follows.

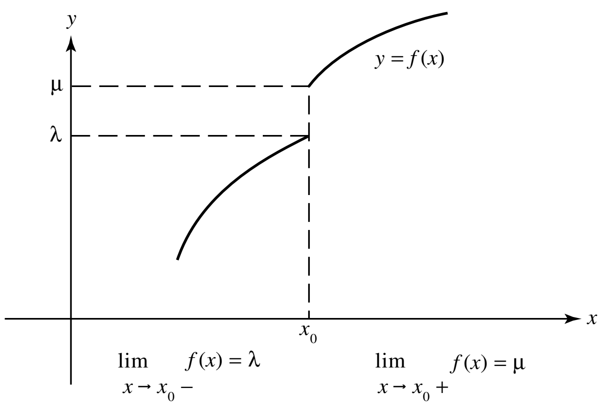


**Figure 2.1.2**

### Definition 2.1.5

**(a)** We say that $f(x)$ *approaches the left-hand limit $L$ as $x$ approaches $x_0$ from the left*, and write

$$\lim_{x\to x_0-} f(x) = L,$$

if $f$ is defined on some open interval $(a, x_0)$ and, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x_0 - \delta < x < x_0.$$

**(b)** We say that $f(x)$ *approaches the right-hand limit* $L$ *as* $x$ *approaches* $x_0$ *from the right*, and write

$$\lim_{x\to x_0+} f(x) = L,$$

if $f$ is defined on some open interval $(x_0, b)$ and, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x_0 < x < x_0 + \delta. \qquad \blacksquare$$

Figure 2.1.2 shows the graph of a function that has distinct left- and right-hand limits at a point $x_0$.

**Example 2.1.8** Let

$$f(x) = \frac{x}{|x|}, \quad x \ne 0.$$

If $x < 0$, then $f(x) = -x/x = -1$, so

$$\lim_{x\to 0-} f(x) = -1.$$

If $x > 0$, then $f(x) = x/x = 1$, so

$$\lim_{x\to 0+} f(x) = 1. \qquad \blacksquare$$

**Example 2.1.9** Let

$$g(x) = \frac{x + |x|(1+x)}{x} \sin\frac{1}{x}, \quad x \ne 0.$$

If $x < 0$, then

$$g(x) = -x \sin\frac{1}{x},$$

so

$$\lim_{x\to 0-} g(x) = 0,$$

since

$$|g(x) - 0| = \left| x \sin\frac{1}{x} \right| \le |x| < \epsilon$$

if $-\epsilon < x < 0$; that is, Definition 2.1.5**(a)** is satisfied with $\delta = \epsilon$. If $x > 0$, then

$$g(x) = (2+x)\sin\frac{1}{x},$$

which takes on every value between $-2$ and $2$ in every interval $(0, \delta)$. Hence, $g(x)$ does not approach a right-hand limit at $x$ approaches 0 from the right. This shows that a function may have a limit from one side at a point but fail to have a limit from the other side. $\blacksquare$

**Example 2.1.10** We leave it to you to verify that

$$\begin{aligned}\lim_{x\to 0+}\left(\frac{|x|}{x}+x\right) &= 1,\\ \lim_{x\to 0-}\left(\frac{|x|}{x}+x\right) &= -1,\\ \lim_{x\to 0+} x\sin\sqrt{x} &= 0,\end{aligned}$$

and $\lim_{x\to 0-}\sin\sqrt{x}$ does not exist. ■

Left- and right-hand limits are also called *one-sided limits.* We will often simplify the notation by writing

$$\lim_{x\to x_0-} f(x) = f(x_0-) \quad \text{and} \quad \lim_{x\to x_0+} f(x) = f(x_0+).$$

The following theorem states the connection between limits and one-sided limits. We leave the proof to you (Exercise 12).

**Theorem 2.1.6** *A function $f$ has a limit at $x_0$ if and only if it has left- and right-hand limits at $x_0$, and they are equal. More specifically,*

$$\lim_{x\to x_0} f(x) = L$$

*if and only if*

$$f(x_0+) = f(x_0-) = L.$$

With only minor modifications of their proofs (replacing the inequality $0 < |x-x_0| < \delta$ by $x_0-\delta < x < x_0$ or $x_0 < x < x_0+\delta$), it can be shown that the assertions of Theorems 2.1.3 and 2.1.4 remain valid if "$\lim_{x\to x_0}$" is replaced by "$\lim_{x\to x_0-}$" or "$\lim_{x\to x_0+}$" throughout (Exercise 13).

## Limits at $\pm\infty$

Limits and one-sided limits have to do with the behavior of a function $f$ near a limit point of $D_f$. It is equally reasonable to study $f$ for large positive values of $x$ if $D_f$ is unbounded above or for large negative values of $x$ if $D_f$ is unbounded below.

**Definition 2.1.7** We say that $f(x)$ *approaches the limit $L$ as $x$ approaches* $\infty$, and write

$$\lim_{x\to\infty} f(x) = L,$$

if $f$ is defined on an interval $(a,\infty)$ and, for each $\epsilon > 0$, there is a number $\beta$ such that

$$|f(x)-L| < \epsilon \quad \text{if} \quad x > \beta.$$ ■

Figure 2.1.3 provides an illustration of the situation described in Definition 2.1.7.

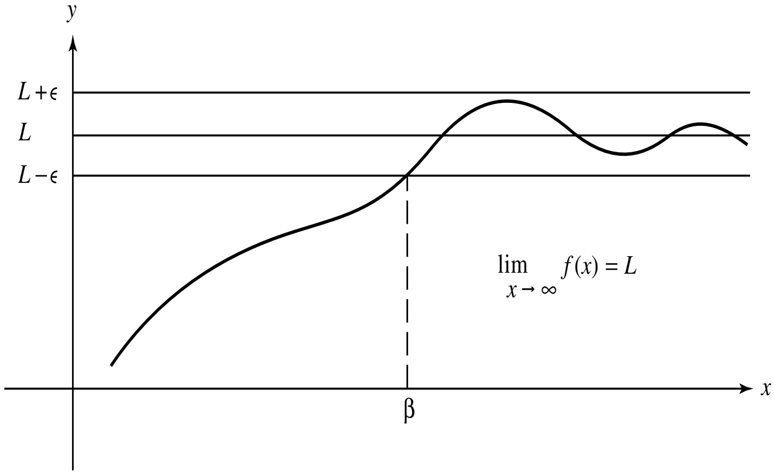


**Figure 2.1.3**

We leave it to you to define the statement "$\lim_{x\to-\infty} f(x) = L$" (Exercise 14) and to show that Theorems 2.1.3 and 2.1.4 remain valid if $x_0$ is replaced throughout by $\infty$ or $-\infty$ (Exercise 16).

**Example 2.1.11** Let

$$f(x) = 1 - \frac{1}{x^2}, \quad g(x) = \frac{2|x|}{1+x}, \quad \text{and} \quad h(x) = \sin x.$$

Then

$$\lim_{x\to\infty} f(x) = 1,$$

since

$$|f(x) - 1| = \frac{1}{x^2} < \epsilon \quad \text{if} \quad x > \frac{1}{\sqrt{\epsilon}},$$

and

$$\lim_{x\to\infty} g(x) = 2,$$

since

$$|g(x) - 2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \epsilon \quad \text{if} \quad x > \frac{2}{\epsilon}.$$

However, $\lim_{x\to\infty} h(x)$ does not exist, since $h$ assumes all values between $-1$ and 1 in any semi-infinite interval $(\tau, \infty)$.

We leave it to you to show that $\lim_{x\to-\infty} f(x) = 1$, $\lim_{x\to-\infty} g(x) = -2$, and $\lim_{x\to-\infty} h(x)$ does not exist (Exercise 17). ■

We will sometimes denote $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ by $f(\infty)$ and $f(-\infty)$, respectively.

## Infinite Limits

The functions

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2}, \quad p(x) = \sin\frac{1}{x},$$

and

$$q(x) = \frac{1}{x^2}\sin\frac{1}{x}$$

do not have limits, or even one-sided limits, at $x_0 = 0$. They fail to have limits in different ways:

- $f(x)$ increases beyond bound as $x$ approaches 0 from the right and decreases beyond bound as $x$ approaches 0 from the left;
- $g(x)$ increases beyond bound as $x$ approaches zero;
- $p(x)$ oscillates with ever-increasing frequency as $x$ approaches zero;
- $q(x)$ oscillates with ever-increasing amplitude and frequency as $x$ approaches 0.

The kind of behavior exhibited by $f$ and $g$ near $x_0 = 0$ is sufficiently common and simple to lead us to define *infinite limits*.

**Definition 2.1.8** We say that $f(x)$ *approaches* $\infty$ *as* $x$ *approaches* $x_0$ *from the left*, and write

$$\lim_{x\to x_0-} f(x) = \infty \quad \text{or} \quad f(x_0-) = \infty,$$

if $f$ is defined on an interval $(a, x_0)$ and, for each real number $M$, there is a $\delta > 0$ such that

$$f(x) > M \quad \text{if} \quad x_0 - \delta < x < x_0. \qquad \blacksquare$$

**Example 2.1.12** We leave it to you to define the other kinds of infinite limits (Exercises 19 and 21) and show that

$$\lim_{x\to0-}\frac{1}{x} = -\infty, \quad \lim_{x\to0+}\frac{1}{x} = \infty;$$
$$\lim_{x\to0-}\frac{1}{x^2} = \lim_{x\to0+}\frac{1}{x^2} = \lim_{x\to0}\frac{1}{x^2} = \infty;$$
$$\lim_{x\to\infty} x^2 = \lim_{x\to-\infty} x^2 = \infty;$$

and

$$\lim_{x\to\infty} x^3 = \infty, \quad \lim_{x\to-\infty} x^3 = -\infty. \qquad \blacksquare$$

Throughout this book, "$\lim_{x\to x_0} f(x)$ exists" will mean that

$$\lim_{x\to x_0} f(x) = L, \quad \text{where } L \text{ is } \textit{finite}.$$

**Example 2.1.15** Let

$$g(x) = \frac{2x^2 - x + 1}{3x^2 + 2x - 1}.$$

Trying to find $\lim_{x\to\infty} g(x)$ by applying a version of Theorem 2.1.4 to this fraction as it is written leads to an indeterminate form (try it!). However, by rewriting it as

$$g(x) = \frac{2 - 1/x + 1/x^2}{3 + 2/x - 1/x^2}, \quad x \neq 0,$$

we find that

$$\lim_{x\to\infty} g(x) = \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} 1/x + \lim_{x\to\infty} 1/x^2}{\lim_{x\to\infty} 3 + \lim_{x\to\infty} 2/x - \lim_{x\to\infty} 1/x^2} = \frac{2-0+0}{3+0-0} = \frac{2}{3}. \qquad \blacksquare$$

## Monotonic Functions

A function $f$ is *nondecreasing* on an interval $I$ if

$$f(x_1) \le f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2, \tag{19}$$

or *nonincreasing* on $I$ if

$$f(x_1) \ge f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2. \tag{20}$$

In either case, $f$ is *monotonic* on $I$. If $\le$ can be replaced by $<$ in (19), $f$ is *increasing* on $I$. If $\ge$ can be replaced by $>$ in (20), $f$ is *decreasing* on $I$. In either of these two cases, $f$ is *strictly monotonic* on $I$.

**Example 2.1.16** The function

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 2, & 1 \le x \le 2, \end{cases}$$

is nondecreasing on $I = [0, 2]$ (Figure 2.1.4), and $-f$ is nonincreasing on $I = [0, 2]$.

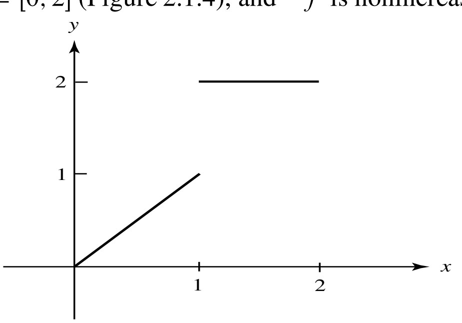

**Figure 2.1.4**

The function $g(x) = x^2$ is increasing on $[0, \infty)$ (Figure 2.1.5),

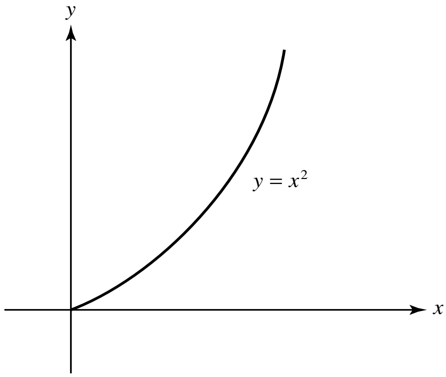


**Figure 2.1.5**

and $h(x) = -x^3$ is decreasing on $(-\infty, \infty)$ (Figure 2.1.6). ■

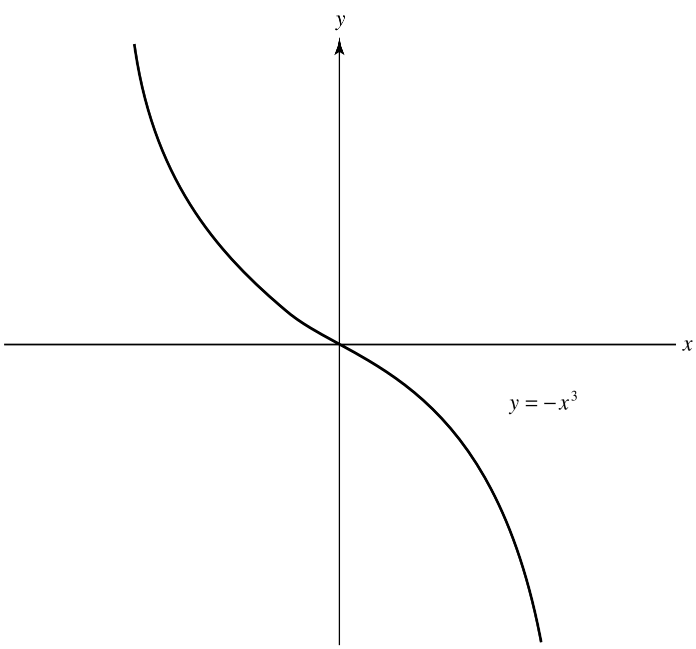


**Figure 2.1.6**